IMPLICATIONS OF THE R-MODE INSTABILITY OF ROTATING RELATIVISTIC STARS

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Several recent surprises appear dramatically to have improved the likelihood that the spin of rapidly rotating, newly formed neutron stars (and, possibly, of old stars spun up by accretion) is limited by a nonaxisymmetric instability driven by gravitational waves. Except for the earliest part of the spin-down, the axial l=m=2 mode (an r-mode) dominates the instability, and the emitted waves may be observable by detectors with the sensitivity of LIGO II. A review of these hopeful results is followed by a discussion of constraints on the instability set by dissipative mechanisms, including viscosity, nonlinear saturation, and energy loss to a magnetic field driven by differential rotation.

1 Introduction

We review here recent work on a gravitational-wave driven instability that may sharply limit the spin of young, rapidly rotating neutron stars. Andersson and Kokkotas[5] have given a comprehensive review of recent work on this r-mode instability and its physical implications, following several earlier reviews[17,39,73]. The last of these, a general review of the gravitational-radiation driven instability, is more pedagogical and not limited to work on r-modes. Included in the present review is a survey of recent work on nonlinear saturation and mode coupling not included in these earlier articles.

The first of the surprises mentioned in the abstract was the discovery that the r-modes, rotationally restored modes that have axial parity for spherical models, are unstable in perfect fluid models with arbitrarily slow rotation. Indicated in numerical work by Andersson [1], the instability is implied in a nearly Newtonian context by the Newtonian expression for the r-mode frequency [59,1,19], and a computation by Friedman and Morsink [19] of the canonical energy of initial data showed (independent of assumptions about the existence of discrete modes) that the instability is a generic feature of axial-parity perturbations of relativistic stars. Studies of the viscous and radiative timescales associated with the r-modes (Lindblom et al.[44], Owen et al. [58], Andersson et al. [6], Kokkotas and Stergioulas [30], Lindblom et al. [43]) revealed a second surprising result: The growth time of r-modes driven by current-multipole gravitational radiation is significantly shorter than had been expected, so short, in fact, that the instability to gravitational radiation reaction easily dominates viscous damping in hot, newly formed neutron stars (see Fig. 5 below). As a result, a neutron star that is rapidly rotating at birth now appears likely to spin down by radiating most of its angular momentum in gravitational waves. (See, however, the caveats below.)

Nearly simultaneous with these theoretical surprises was the discovery by Marshall et.al. [53] of a fast (16ms) pulsar in a supernova remnant (N157B) in the Large Magellanic Cloud. From the pulsar's period and period derivative, and from the estimated age of the remnant, the initial period is estimated at less than 10ms, implying a class of neutron stars that are rapidly rotating at birth. Fortifying this conclusion is the belief that accretion-induced collapse of O-Ne-Mg and C-O dwarfs leads to rapidly rotating neutron stars [57] and the likelihood that magnetars are rapidly rotating at birth.

Spurred by these surprises, over fifty authors have worked on aspects of the r-mode instability:

- Examining the modes themselves, for Newtonian and relativistic models. [1,2,8,25,26,27,28,29,30,40,43,47,48,49,50,66,79,80,82,83,85]

- Studying the waveforms and detectability of the gravitational waves they emit [6,7,11,15,22,38,44,51,52,58,62,70]

- Finding the dominant mechanisms of the effective shear and bulk viscosities, including effects of a superfluid interior and a solid crust. [3,4,6,10,34,36,42,45,54,61,65,78,84]

- Finding the maximum mode amplitude permitted by nonlinear fluid evolution. [18,35,46,69,75]

- Finding the nonlinear differential rotation associated with r-modes and asking whether such rotation dissipates in a magnetic field the energy of an unstable r-mode. [18,23,35,63,72]

2   The Gravitational-Wave Driven Instability

All rotating perfect fluid stars are subject to a nonaxisymmetric instability driven by gravitational radiation. The instability was found by Chandrasekhar [13] for the $l = m = 2$ polar mode of the uniform-density, uniformly rotating Maclaurin spheroids. Although this mode is unstable only for rapidly rotating models, by looking at the canonical energy of initial data with arbitrary values of m, Friedman and Schutz [20] and Friedman [16] showed that the instability is a generic feature of rotating perfect fluid stars, that even slowly rotating perfect-fluid models are formally unstable.

For a normal mode of the form $e^{i(\sigma t + m\varphi)}$ this nonaxisymmetric instability acts in the following manner: In a non-rotating star, gravitational radiation removes positive angular momentum from a forward moving mode and negative angular momentum from a backward moving mode, thereby damping all time-dependent, non-axisymmetric modes. In a star rotating sufficiently fast, however, a backward moving mode can be dragged forward as seen by an inertial observer; and it will then radiate positive angular momentum. The angular momentum of the mode, however, remains negative, because the perturbed star has lower total angular momentum than the unperturbed star. As positive angular momentum is removed from a mode with negative angular momentum, the angular momentum of the

mode becomes increasingly negative, implying that its amplitude increases: The mode is driven by gravitational radiation.

The conclusion, that a mode is unstable if it is prograde relative to infinity and retrograde relative to the star is equivalent to requiring that its frequency satisfies the condition,

$$\sigma(\sigma + m\Omega) < 0. \tag{1}$$

For the polar f- and p-modes, the frequency is large and approximately real. Condition (1) will be met only if $|m\Omega|$ is of order $|\sigma|$, so that for a given angular velocity the instability will set in first through modes with large m.

3   Unstable r-Modes

Part of the reason that axial modes (r-modes) were not studied extensively for neutron stars is that for spherical stars they are stationary, convective currents. [a] That is, axial perturbations belonging to an l, m representation of the rotation group behave under parity as $(-1)^{l+1}$, opposite to Y_{lm}. Because any scalar can be written as a sum of Y_{lm}, no scalar perturbation of a spherical star has axial parity. Axial perturbations have, to order Ω^2, vanishing $\delta p, \delta \rho$ and $\delta \Phi$. Only their velocity perturbation is nonzero, and it has the form

$$A r^l \mathbf{r} \times \nabla Y_{lm}. \tag{2}$$

A diagram of the velocity field (due to L. Lindblom) for $l = m = 2$ is reproduced below.

The fluid elements move along the integral curves of $\mathbf{v} + \delta \mathbf{v}$, ellipses, to first order in the perturbation, as shown in Fig. 2. The restoring force in a rotating frame can be regarded as the Coriolis force, leading to a frequency of oscillation proportional to angular velocity of the star.

At the onset of instability, a mode's frequency vanishes. For f-modes and p-modes (both polar modes) with low values of l, the frequency of a mode is of order $\sigma \sim 1/t_{\text{dynamical}} \sim \sqrt{G\rho}$, implying instability points at similarly large values of the star's angular velocity.[56,73] That is, to drag a mode that is retrograde relative to the fluid forward relative to infinity, one must have $\Omega \sim \sigma_{\text{sphericalstar}}/l$. In fact, when viscosity is taken into account, polar modes are estimated to be unstable in a uniformly rotating star only for $\Omega > 0.9\Omega_K$, where Ω_K is the Kepler limit on a star's rotation, the angular velocity of a satellite at its equator. [24,14,37,41] [b] The frequency of an r-mode, however is already zero for a spherical star; and *for any nonzero rotation, every r-mode of a perfect-fluid model is unstable*.[1,19] The instability in models of slowly rotating, nearly Newtonian stars, follows from the

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[a] This statement, although formally true, is somewhat misleading. A solution with axial parity to the linearized Euler equation *is* a stationary current, but the nonlinear terms in the Euler equation are important once a fluid element has moved a distance of order the radius of the star, and the nonlinear solution is time-dependent.

[b] This estimate may decrease, perhaps to $0.8 \ \Omega_K$ when a general-relativistic computation with viscosity is carried out.

Flow Pattern for the $\mathbf{m=2}$ r-mode
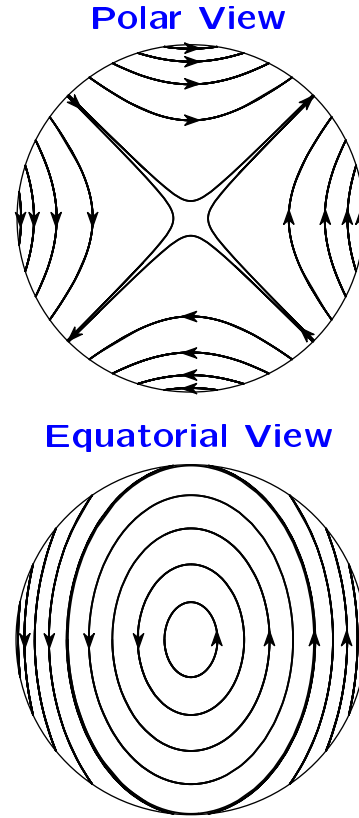
Polar View

Equatorial View

Figure 1. The perturbed velocity field of the $l = m = 2$ r-mode.

fact that the frequencies [59] satisfy the criterion (1),

$$\sigma(\sigma + m\Omega) = -\frac{2(l-1)(l+2)m^2\Omega^2}{l^2(l+1)^2} < 0. \tag{3}$$

As noted below, it is likely, but not yet certain that relativistic stars generically have a discrete spectrum of r-modes; but a computation by Friedman and

Figure 2. Fluid trajectories of the l=m=2 r-mode

Morsink [19] of the canonical energy of initial data is independent of the existence
of normal modes, and it shows that the instability is a generic feature of
axial-parity fluid perturbations of relativistic stars.

For stars with general equations of state, the r-modes describe the dynamical
evolution of initial perturbations that have axial parity.[59,67,71] In barotropic
Newtonian models, however, the only purely axial modes allowed are the r-modes
with $l = m$ and simplest radial behavior.[60] The disappearance of the purely axial
modes with $l \neq m$ occurs for the following reason.[48] Axial perturbations of a
spherical star are time-independent convective currents with vanishing perturbed
pressure and density. In spherical *barotropic* stars, stars for which both star and
perturbation are governed by a single one-parameter equation of state, the
gravitational restoring forces that give rise to the g-modes vanish, and they, too,
become time-independent convective currents with vanishing perturbed pressure
and density. Thus, the space of zero frequency modes, which generally consists
only of the axial r-modes, expands for spherical barotropic stars to include the
polar g-modes. This large degenerate subspace of zero-frequency modes is split by
rotation to zeroth order in the star's angular velocity, and the corresponding
modes of rotating barotropic stars are generically hybrids whose spherical limits
are hybrids of axial and polar perturbations. Because their restoring force is
rotational (Coriolis), we have referred both to them and to the r-modes as
rotational modes, and they are called "inertial modes" in the fluid-dynamics
literature.[21]

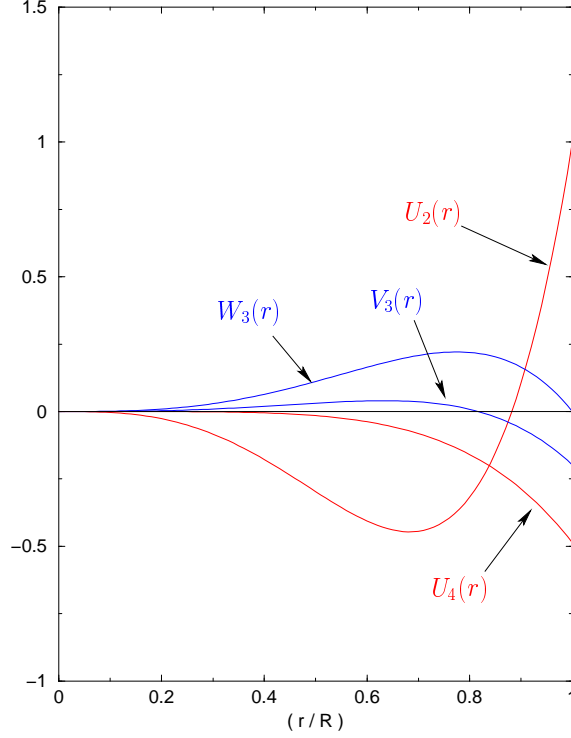To compute these "hybrid" rotational modes, Lockitch and Friedman [48] expand

Figure 3. An r-g "hybrid" rotational mode of a uniform density Newtonian star. The functions shown are coefficients of the spherical harmonic expansion (4) of the perturbed fluid velocity.

the perturbed 3-velocity, δv^a, in vector spherical harmonics (see also Ref. [33]),

$$\delta v^a = \sum_{l=m}^{\infty} \left\{ \frac{1}{r} W_l Y_l^m \nabla^a r + V_l \nabla^a Y_l^m - i U_l \epsilon^{abc} \nabla_b Y_l^m \nabla_c r \right\} e^{i\sigma t}, \qquad (4)$$

and solve the order Ω perturbation equations for the coefficients $U_l(r)$ of the axial-parity terms and the coefficients $W_l(r)$ and $V_l(r)$ of the polar-parity terms. These coefficients are shown in Fig. 3 for a particular hybrid mode of a uniform density Newtonian model. For a pure r-mode, only one of the coefficients $U_l(r)$ would be nonzero.

R-modes of rapidly (and differentially) rotating Newtonian polytropes have been recently computed by Karino et al. [26] and references to the earlier Newtonian literature are given in the Introduction's bibliography above.

3.1   r-Modes of Relativistic Stars

The r-modes of rotating relativistic stars were studied for the first time only recently.[1,8,27,28,29,47,49,66,75,79] As in the Newtonian case,[33,48] a spherical
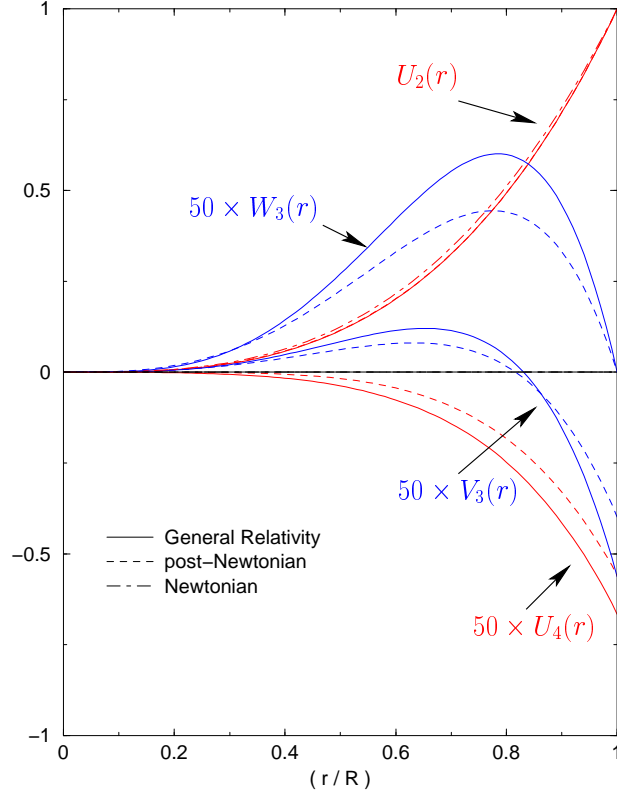
Figure 4. Relativistic corrections to the $l = m = 2$ r-mode of a uniform density star of compactness $2M/R = 0.316$. In Newtonian gravity, only the coefficient $U_2(r)$ is nonzero, i.e. the mode is a pure r-mode. However, the relativistic corrections turn the mode into a hybrid.

barotropic relativistic star has a large degenerate subspace of zero-frequency modes consisting of the axial-parity r-modes and the polar-parity g-modes. Although barotropic Newtonian stars retain a vestigial set of purely axial $l = m$ modes, *rotating relativistic stars of this type have no pure r-modes*,[47,49] no modes whose limit for a spherical star is purely axial. Instead, the Newtonian r-modes with $l = m \geq 2$ acquire relativistic corrections with both axial and polar parity to become discrete rotational modes of the corresponding relativistic models. Relativistic modes have been computed analytically in a post-Newtonian approximation[49] and numerically for slowly rotating polytropes.[50,66,79] (See Fig. 4.) They have also been studied in numerical time evolutions of rapidly rotating relativistic stars using the Cowling approximation (hydro evolution with a frozen spacetime metric).[75] Preliminary results suggest that the growth timescale of the most unstable mode is largely unaffected by the relativistic corrections,[50] differing from the post-Newtonian estimates[44,6] by $\lesssim 10\%$.
In the slow-rotation approximation in which they have so far been studied, axial

perturbations of non-barotropic stars include, remarkably, a *continuous* spectrum. Kojima [27] shows that the axial modes are described by a single, second-order ODE for the modes' radial behavior. (See also the contributions of Kojima and Hosonuma to the MG9 r-mode session, APT7). He argues that the continuous spectrum is implied by the vanishing of the coefficient of the highest derivative term of this equation at some value of the radial coordinate, and Beyer and Kokkotas[8] make the claim precise. As the latter authors point out, the continuous spectrum they find may be an artifact of the vanishing of the imaginary part of the frequency in the slow rotation limit. (Or, more broadly, it may be an artifact of the slow rotation approximation.)

Furthermore, it is possible find discrete normal modes for non-barotropic models. This is straightforward in the case of models for which the coefficient in the Kojima eqation is nonzero within the fluid. (Non-barotropic uniform density models have this property and their r-modes have been computed by a number of groups.)[2,49,66,79] On the other hand, it is not so straightforward when the coefficient of Kojima's equation vanishes within the fluid - as is the case for certain polytropes with realistic compactness. Ruoff and Kokkotas[66] and Yoshida[79] have argued that a discrete r-mode simply does not exist for such models. However, it would be surprising if a small change in, say, the compactness of the star could lead to such a drastic change in the star's physics (the disappearance of the r-modes). It is perhaps more likely that the slow-rotation approximation is breaking down in the vicinity of the singular point and that if one were to regularize the Kojima equation in this region by including terms that are higher order in Ω, one would be able to find discrete r-mode solutions.[2,50]

4   Spin-down and Gravitational Radiation

Early work suggested that the r-mode instability may sharply limit the spin of newly formed, rapidly rotating neutron stars. The radiation may carry off almost all of the star's initial rotational energy amounting to few percent of its rest mass, and the waves may be detectable by LIGO II[44,6,58,11] from events out to 20 Mpc. The spin-down model on which these tantalizing estimates were based assumed that the most unstable r-mode (with multipole indices $l = m = 2$) would be able to grow to an amplitude of order unity before being saturated by some sort of nonlinear process, and, as we will see, that assumption has been supported by subsequent work.

Spin-down computations in the context of linear perturbation theory describe a competition between viscosity and gravitational radiation. In these calculations,[44,58,6,43] the growth time τ of an unstable mode (or the damping time of a mode stabilized by a viscosity) has the form,

$$\frac{1}{\tau} = \frac{1}{\tau_{\mathrm{GR}}} + \frac{1}{\tau_{\mathrm{shearviscosity}}} + \frac{1}{\tau_{\mathrm{bulkviscosity}}}, \tag{5}$$

with τ the e-folding time for each process. (The analysis sketched below can be found in Lindblom et al[44]; see also Ipser and Lindblom[24]). When the energy

radiated per cycle is small compared to the energy of the mode, the imaginary part of the mode frequency is accurately approximated by the expression

$$\frac{1}{\tau} = -\frac{1}{2E}\frac{dE}{dt},$$

(6)

where E is the energy of the mode as measured in the rotating frame,

$$E = \frac{1}{2}\int\left[\rho\delta v^a\delta v_a^* + \left(\frac{\delta p}{\rho} + \delta\Phi\right)\delta\rho^*\right]d^3x.$$

(7)

We have,

$$\frac{dE}{dt} = -\sigma(\sigma + m\Omega)\sum_{l\geq 2}N_l\sigma^{2l}\left(|\delta D_{lm}|^2 + |\delta J_{lm}|^2\right)$$

$$-\int\left(2\eta\delta\sigma^{ab}\delta\sigma_{ab}^* + \zeta\delta\theta\delta\theta^*\right),$$

(8)

where the dissipation due to gravitational radiation [76] has coupling constant

$$N_l = \frac{4\pi G}{c^{2l+1}}\frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2};$$

(9)

$\delta\sigma_{ab}$ and $\delta\theta$ are the coefficients of shear and bulk viscosity; and estimates[14,68] of corresponding coefficients η and ζ are

$$\eta = 2\times 10^{18}\left(\frac{\rho}{10^{15}\text{g}\cdot\text{cm}^{-3}}\right)^{\frac{9}{4}}\left(\frac{10^9 K}{T}\right)^2\ \text{g}\cdot\text{cm}^{-1}\cdot\text{s}^{-1},$$

(10)

and

$$\zeta = 6\times 10^{25}\left(\frac{1\text{Hz}}{\sigma + m\Omega}\right)^2\left(\frac{\rho}{10^{15}\text{g}\cdot\text{cm}^{-3}}\right)^2\left(\frac{T}{10^9 K}\right)^6\ \text{g}\cdot\text{cm}^{-1}\cdot\text{s}^{-1};$$

(11)

Polar and axial radiation arise, respectively, from mass and current multipoles, D_{lm} and J_{lm}, given by the equations,

$$D_{lm} = \int dV r^l\rho Y_{lm}^* \qquad\qquad J_{lm} = \frac{2}{(l+1)c}\int dV r^l\rho\mathbf{v}\cdot\mathbf{r}\times\nabla Y_{lm}^*$$

(12)

The additional factor of v in the current multipoles implies an additional factor of v^2 in the radiated energy of axial modes, and hence a smaller expected rate of radiation for the same multipole: For a mode of amplitude $A = $ (displacement of fluid element)$/R$, with R the stellar radius, we have

$$\frac{dE}{dt}\sim A^2 M^2 R^{2l}\sigma^{2l+2} \qquad\qquad \frac{dE}{dt}\sim A^2 M^2 R^{2l+2}\sigma^{2l+4}.$$

(13)

The extra factor of σ^2 in dE/dt for the current multipoles would make polar modes dominant if it were not for the fact that their frequencies are small when they are unstable. If a newborn neutron star rotates at nearly its maximum frequency it is likely initially to be unstable to both polar and axial modes. Most of its spin-down, however, should be dominated by the $l = m = 2$ r-mode. An optimistic diagram of this spin-down is shown below in Fig. 5. (This figure,
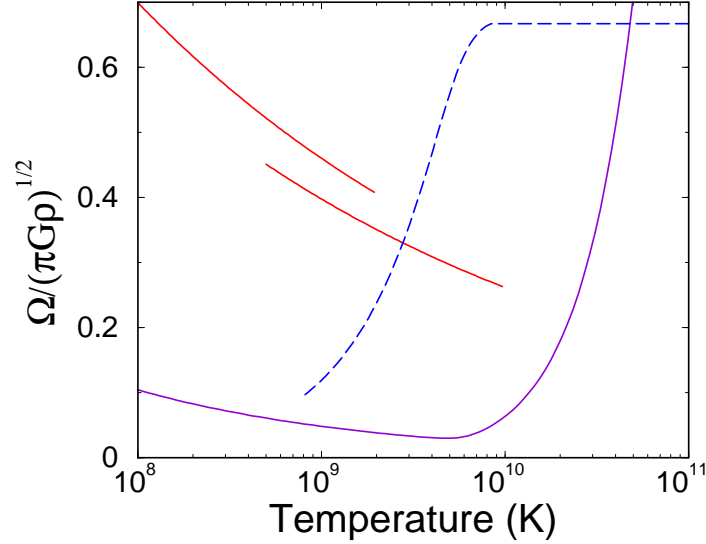
Figure 5. Critical angular velocity vs. temperature for an $n = 1.0$ polytrope. Above the solid (purple) curve, the star rotates rapidly enough for its fastest growing ($l = m = 2$) r-mode to be unstable, whereas below the curve all modes are damped by viscosity. The dashed curve shows the evolution of a rapidly rotating neutron star as it cools and spins down due to the emission of gravitational waves. The red curves show the larger critical angular velocities required for instability when a crust is present, the upper red curve corresponding to the viscosity when a superfluid is present.

prepared by Ben Owen, revises a similar figure in Lindblom, Owen and Morsink [44]) The perturbation here is assumed to reach saturation with an amplitude of order unity, while still maintaining the character of a linear r-mode.

The solid (purple) curve shows, for each temperature T, the minimum angular velocity above which gravitational radiation can dominate viscosity and drive an r-mode instability. Below the solid curve on the right, bulk viscosity due to neutrino production in URCA reactions damps the instability. Below the same solid curve on the left, shear viscosity damps the instability. The upper red curves exhibit the larger shear viscosity from a laminar boundary layer when a crust is present (see Sec. 4.1 below). A newborn star that starts with angular velocity Ω_K follows the dashed trajectory when no crust is present, becoming unstable when the temperature drops below about $3 \times 10^{10} K$. The mode quickly reaches its saturation amplitude and then radiates the star's angular momentum in gravitational waves. Finally, as the star cools and spins down, shear viscosity damps the instability. In the diagram, this occurs at a temperature of about $10^9 K$ (after about a year, with standard cooling), and at an angular velocity below $0.2\Omega_K$. As noted in Sect. 5, the amplitude may grow large enough that the later part of the nonlinear evolution no longer resembles this linear model. This initial scenario is likely to overestimate the duration of the instability and
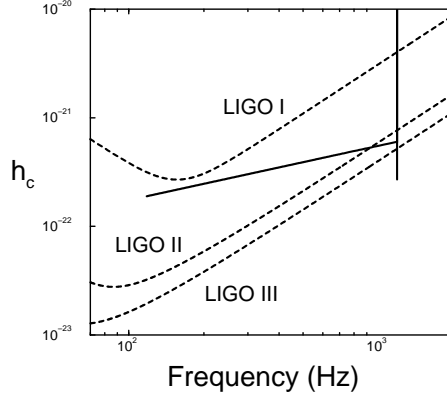
Figure 6. Characteristic gravitational wave amplitude h_c (solid curve) compared to the noise amplitude $h_{\rm rms}$ (dashed curves) for LIGO, assuming perfect knowledge of the r-mode waveform.

the amount of energy radiated, and we discuss below the principal corrections that have been considered.

If the r-modes saturate near unity, a perfect understanding of their waveform would allow a high signal to noise ratio for sources beyond 20 Mpc for detectors with the sensitivity of LIGO II, as dramatized by Fig. 6 (taken from Owen et al. [58] with permission of the author).

In reality, however, the large uncertainty in the waveform substantially reduces the prospects for detection in the immediate future. A study by Brady and T. Creighton [11], following the work by Owen et al[58], sets a lower limit on detectability by assuming no knowledge of the source, finding that newly formed neutron stars should be detectable by LIGO II with narrow banding out to about 8 Mpc, with uncertainty allowing a range of perhaps 4-20 Mpc. The Virgo cluster would then likely be out of reach, and r-modes could not be detected until more sensitive detectors were available.

A stochastic background of gravitational waves produced by a cosmological population of newly formed neutron stars is also likely to be detected only by interferometers with advanced sensitivities (Owen et al. [58], Ferrari et al. [15], Schneider et al. [70]).

4.1 Role of a Solid Crust

Beginning with Bildsten and Ushormirsky,[10] several authors have considered a large increase in shear viscosity in a boundary layer near a solid crust.[4,65,36,45,78,54] Because the melting temperature of the crust is estimated to be 10^{10}K, with a factor 2 uncertainty, a crust may alter the r-mode instability of newly formed stars and will certainly be important in any r-mode instability of old stars spun up by accretion.

If the mode's velocity field vanishes in the crust, it will fall rapidly to zero in a boundary layer. In laminar flow,[32] the thickness of the boundary layer (*Eckman*

layer),

$$d = \sqrt{\frac{\nu}{2\Omega}}, \tag{14}$$

can be estimated by equating the acceleration $\delta[(\partial_t v \cdot) \nabla v)] \sim \Omega \delta v$ of a fluid element to the viscous force per unit mass, $\sim \eta \delta v / \rho d^2$. For a neutron star above the superfluid transition temperature, the coefficient of viscosity η is of order $\eta = 2 \times 10^{18} \rho_{15}^{9/4} T_9^{-2}$ gm/cm-s, and d is a few cm.

As this rough estimate suggests, when a crust is present, dissipation in the viscous boundary layer dominates viscous energy loss in Eq. (8): Because the integrand $\eta \delta \sigma^{ab} \delta \sigma_{ab}$ in the layer is enhanced by a factor of order R^2/d^2 (R the radius of the star), the dissipation in the layer is larger than that in the interior by a factor R/d. Estimate of the higher angular velocities needed for instability are shown as the red curves in Fig. 5.

Levin and Ushomirsky[36] point out that this conclusion somewhat overstates the case, because the crust will participate in an r-mode. Detailed calculations by Yoshida and Lee [84] describe the interaction of the stellar r-modes with r-modes of the crust, finding a series of avoided crossings. Levin and Ushomirsky find that the avoided crossings lead to a wide variation in the fractional drop in δv across the boundary layer, which can range from 1/20 to 1, allowing a possible decrease in the boundary layer dissipation by a factor $(1/20)^2 = 1/400$, compared to the estimate above.

These estimates are based on an assumption of laminar flow, but Wu et al[78] show that in a maximally rotating star, a mode whose amplitude exceeds $\sim 10^{-3}$ will become turbulent. Because the relevant amplitudes are larger than this, theirs is the best current computation of viscous damping in the presence of a crust.

The high boundary-layer viscosity may not significantly alter an r-mode instability of newborn neutron stars, because the $l = m = 2$ r-mode is likely to be unstable by the time the temperature has dropped to between $2 \times 10^{10} K$ and $4 \times 10^{10} K$, slightly above the temperature at which the crust is expected to solidify. If this is the case, Lindblom, Owen, and Ushomirsky[45] show that the r-mode will prevent a solid crust from forming. Instead, they argue, chunks of ice will lie in the outer part of star, with a density adjusted to make the heat dissipated by the ice flow balance the mode's gravitational-wave driven rate of energy increase. If, as expected, the mode amplitude is above 10^{-4} before a crust forms, the ice flow will apparently not significantly affect the r-mode spin-down.

4.2   Instability in Old Stars Spun-Up by Accretion?

For old neutron stars spun up by accretion, the crust is likely to play a crucial role. Wagoner's hope[77] was that accreting neutron stars would spin up until they were unstable (to polar modes, in his paper) and would then radiate angular momentum in gravitational waves at a rate that balanced the angular momentum gained in accretion. Bildsten [9] and Andersson, Kokkotas, and Stergioulas [7] considered the mechanism for r-modes in low-mass x-ray binaries (LMXBs), the latter suggesting that it might account for the narrow range of observed angular velocities.

Brown and Ushomirsky [12], however, concluded that low observed luminosities were inconsistent with the Wagoner mechanism. Levin, [34] reexamined the mechanism for r-modes, finding a cycle in which the star is spun up to instability, but reaches no equilibrium. Instead, by heating up the star, the mode lowers the viscous damping, becomes increasingly unstable, and radiates its energy in a short time (less than a year). It is then gradually spun up (for more than 10^6 years), and the cycle repeats. Armed with this revised scenario, Andersson et al[4] suggested that the instability is responsible for the maximum observed angular velocity (1.6 ms) of old neutron stars and is consistent with the observed range of LMXB spins.

Despite the large difference between turbulent boundary-layer viscosity and the shear viscosity of the neutron-star interior that Andersson et al. [4] had used, Wu et al.[78] find similar results; that is, like Andersson et al. [4], they find spin-rates consistent with those of LMXBs. The maximum angular velocity grows with the accretion rate and falls in the narrow range of 490-700 Hz for $\dot{M}/M$ between 10^{-11} and 10^{-8} yr^{-1}, when the crust is rigid. (If the fractional drop in δv across the boundary layer is only 1/10, the maximum angular velocities fall to 220 - 300 Hz; and an unexpectedly low melting temperature for the crust will again reduce the maximum Ω.)

Levin's cycle is qualitatively similar in this analysis, but the spin-down time increases to 10^3 yr, and the spin-up time to 10^7 yr.

In a study of hypercritical accretion flow onto neutron stars, Yoshida and Eriguchi [81] find that an unstable r-mode strongly limits the neutron-star spin. The high energy input associated with accretion is large enough that Levin's cycle is not seen. These are then a class of systems, possible precursors to compact binary systems, in which accreting neutron stars may reach a Wagoner equilibrium.

Finally, we should mention a recent preprint by Mendell, [54] examining the damping of r-modes by a magnetic field in a viscous boundary layer. The damping is large for magnetic fields of order $10^{12}G$, and, like the boundary-layer viscosity, could prevent an r-mode instablity in a newborn star if a crust forms before an unstable mode has time to grow.

5   Nonlinear Calculations

Much of the very recent work on the r-mode instability has addressed the nonlinear evolution of the r-modes. The central issue is whether the instability found in idealized models survives the physics that governs a young neutron star. We discussed the role of the crust in the last section, and now ask two questions related to a mode's nonlinear evolution: Does nonlinear coupling to other modes allow an unstable r-mode to grow to unit amplitude? Does the background star retain a uniform rotation law as it spins down or does a growing r-mode generate significant differential rotation? The importance of this last question was emphasized by Spruit [72] and by Rezzolla, Lamb and Shapiro [63] who argued that differential rotation would wind up a toroidal magnetic field and drain the oscillation energy of the r-mode. (See also Rezzolla's contribution to the MG9

r-mode workshop APT7). A number of different approaches have since been applied to the nonlinear r-mode problem in an attempt to address these questions. One notable approach is the direct numerical evolution of the nonlinear equations describing a self-gravitating fluid. Stergioulas and Font [75] have performed 3-D general relativistic hydrodynamic evolutions in the Cowling approximation, and Lindblom, Tohline and Vallisneri [46] have performed 3-D Newtonian hydrodynamic evolutions with an added driving force representing gravitational radiation-reaction, equivalent to that computed previously by Rezzolla et al. [64] Stergioulas and Font [75] construct an equilibrium model of a rapidly rotating relativistic star and add to it an initial perturbation that roughly approximates its $l = m = 2$ r-mode. They then evolve the perturbed star using the nonlinear hydrodynamic equations with the spacetime metric held fixed to its equilibrium value (the relativistic Cowling approximation). They find no evidence for suppression of the mode on a dynamical timescale, even when the mode amplitude, A, is initially taken to be of order unity. Because of the approximate nature of the initial perturbation, other oscillation modes are excited in the initial data. Recall that for a star with a barotropic equation of state, the generic rotationally restored mode is not a pure axial-parity r-mode, but an r-g "hybrid" mode (Sect. 3). Stergioulas and Font [75] find that a number of these hybrid modes are excited in their initial data with good agreement between the inferred frequencies and earlier results from linear perturbation theory [48]. In their published work, they find no evidence that the dominant mode is leaking its oscillation energy to other modes on a dynamical timescale. Instead, a nonlinear version of an r-mode appears to persist over the time of the run, about 25 rotations of the star. In additional runs with amplitudes substantially larger than unity, however, one no longer sees a coherent r-mode. This may be evidence of nonlinear saturation, but further runs with more accurate initial data will be necessary to conclude this definitively [74]. (See also Stergioulas' contribution to the r-mode workshop APT7).

These conclusions are consistent with preliminary results from studies of nonlinear mode-mode couplings at higher order in perturbation theory [69,55]. Other r-modes of a nonbarotropic star seem to give no indication of a strong coupling to the $l = m = 2$ r-mode unless its amplitude is unphysically large ($A \gtrsim 30$!) [55]. Work is still in progress on the nonlinear coupling of the dominant r-mode to the g-modes of nonbarotropic stars [55] and to the hybrid modes of barotropic stars [69].

The results of Stergioulas and Font [75] have also been confirmed and significantly extended by the calculation of Lindblom, Tohline and Vallisneri. [46] In Stergioulas and Font's calculation the growth of the unstable r-mode does not occur because the spacetime dynamics have been turned off. However, it would be impossible to model this growth anyway even in a fully general relativistic hydrodynamic evolution, because the timescale on which the mode grows due to the emission of gravitational waves far exceeds the dynamical timescale of a rapidly rotating neutron star.

To simulate the growth of the dominant r-mode in a calculation accessible to current supercomputers, Lindblom, Tohline and Vallisneri [46] take a different approach. They begin by constructing an equilibrium model of a rapidly rotating

Newtonian star and add to it a small initial perturbation corresponding to its $l = m = 2$ r-mode. They then evolve the perturbed star by the equations of Newtonian hydrodynamics with a post-Newtonian radiation-reaction force that drives the current quadrupole associated with the $l = m = 2$ r-mode.

By artificially scaling up the strength of the driving force, they are able to shorten the growth time of the unstable r-mode by a factor of 4500. In the resulting simulation the mode grows exponentially from an amplitude $A = 0.1$ to $A = 2.0$ in only about 20 rotations of the star.

With this magnified radiation-reaction force, Lindblom, Tohline and Vallisneri [46] are able to confirm the general features of the simplified r-mode spin-down models [44,6,58,11]. In their simulation, the star begins to spin down noticeably when the amplitude of the dominant mode is of order unity, and ultimately about 40% of the star's angular momentum is radiated away. The evolution of the star's angular momentum as computed numerically agrees well with the predicted angular momentum loss to gravitational radiation. If their model is accurate, however, gravitational radiation would not be emitted steadily at a saturation amplitude, but would die out after saturation and then reappear as the mode regenerates.

Again, there is no evidence of nonlinear saturation for mode amplitudes $A \lesssim 1$. The growth of the mode is eventually suppressed at an amplitude $A \simeq 3.4$, and the amplitude drops off sharply thereafter. Lindblom, Tohline and Vallisneri argue that the mechanism suppressing the mode is the formation of shocks associated with the breaking of surface waves on the star. They find no evidence of mass-shedding, nor of coupling of the dominant mode to the other r-modes or hybrid modes of their Newtonian barotropic model.

These various studies all provide evidence pointing to the same conclusion: the most unstable r-mode appears likely to grow to an amplitude of order unity before being suppressed by nonlinear hydrodynamic processes. It is important to emphasize, however, that the 3-D numerical simulations have probed nonlinear processes occurring only on dynamical timescales and that the actual growth timescale for the r-mode instability is longer by a factor of order 10^4. It is possible that the instability may be suppressed by hydrodynamic couplings occurring on timescales that are longer than the dynamical timescale but shorter than the r-mode growth timescale. Further work clearly needs to be done before definitive conclusions can be drawn. Particularly relevant will be the results from the ongoing mode-mode coupling studies [69,55].

Turning to the question of differential rotation, deviations from a uniform rotation law are observed in both of the 3-D numerical simulations [75,46]. It has been proposed that differential rotation will be driven by gravitational radiation-reaction [72] as well as being associated with the second order motion of the r-mode, itself [63]. In a useful toy model, Levin and Ushomirsky [35] calculated an exact r-mode solution in a thin fluid shell and found both sources of differential rotation to be present.

To address in more detail the issue of whether or not the r-mode instability would generate significant differential rotation, Friedman, Lockitch and Sá [18] have calculated the axisymmetric part of the second order r-mode. We work to second

order in perturbation theory with the equilibrium solution taken to be either a slowly rotating polytrope (with index $n = 1$) or an arbitrarily rotating uniform density star (a Maclaurin spheroid). The first order solution, which appears in the source term of the second order equations, is taken to be a pure $l = m$ r-mode with amplitude A.

We find that differential rotation is indeed generated both by gravitational radiation-reaction and by the quadratic source terms in Euler's equation; however, the latter dominate a post-Newtonian expansion. The functional form of the differential rotation is independent of the equation of state - the axisymmetric, second order change in v^φ being proportional to z^2 (in cylindrical coordinates) for both the polytrope and Maclaurin.

Our result extends that of Rezzolla, Lamb and Shapiro [63] who computed the order A^2 differential drift resulting from the linear r-mode velocity field. These authors neglect the nonlinear terms in the fluid equations and argue (based on an analogy with shallow water waves) that the contribution from the neglected terms might be irrelevant. Indeed, for sound waves and shallow water waves, the fluid drift computed using the linear velocity field turns out to be exact to second order [31]; thus, one may safely ignore the nonlinear terms. However, for the motion of a fluid element associated with the r-modes, we find that there is in fact a non-negligible contribution from the second-order change in v^φ. Interestingly, the resulting second order differential rotation is stratified on cylinders. It remains to be seen whether the coupling of this differential rotation to the star's magnetic field does indeed imply suppression of the r-mode instability.

6   Comments on Future Work

The overarching questions are whether unstable r-modes limit the spin of either young, rapidly rotating neutron stars or of old stars spun up by accretion; and if young stars are unstable, how accurately can we characterize the r-mode's wave form?

The nonlinear studies of r-mode instability are still at an early stage, and it is not yet certain what limits the amplitude of an r-mode. All computations so far agree that there is no saturation until large amplitude is reached, and both numerical groups observe the formation of shocks and surface-wave breaking at large amplitudes. [46,74] Still unclear, however, is whether sudden damping of the mode by wave breaking is an artifact of the numerics or of an amplified radiation reaction; the possibility remains that mode-mode coupling enforces a smoother limit on the r-mode amplitude.

It is similarly unclear whether an r-mode will wind up a magnetic field or whether, in the presence of a background magnetic field, there is a slightly altered mode that does not secularly change the field.

On the mathematical side, deciding whether the spectrum of rotational modes of relativistic stars is continuous or discrete may be a tractable problem and may have implications for neutron-star physics.

More generally, uncertainties in the microphysics (in, e.g., bulk viscosity due to hyperon production in the core, or in the coupling of superfluid and crust) are

large enough that we are unlikely to decide soon whether unstable r-modes play a role in the lives of neutron stars. But a discovery of their gravitational waves could decide the issue for us.

Acknowledgments

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